Matrix Norm

The *norm* of a matrix[1] extends the concept of a vector norm[2] and a measure of the size of a matrix. The norm of a matrix A is denoted $\|A\|$. There are several different types of norms asd the type of norm is indicated by a subscript.

Properties of a Matrix Norm

For any $m \times n$ matrices A and B the norm $\|.\|$ must satisfy the following properties

$$\|A\| \geq 0 \ \ (\text{ and } \|A\| = 0 \ \ \Leftrightarrow \ \ A = 0),$$

$$\|A + B\| \leq \|A\| + \|B\|,$$

$$\|\alpha A\| = |\alpha|\|A\|,$$

where α is a scalar.

1-norm

For any $m \times n$ matrices A, the 1-norm is the maximum column sum, defined as follows:

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

∞-norm

For any $m \times n$ matrices A, the ∞-norm is the maximum row sum, defined as follows:

$$\|A\|_\infty = \max_i \sum_{i=1}^{n} |a_{ij}|$$

Frobenius norm

The Frobenius norm is defined as follows

$$\|A\|_F = \sqrt[2]{\sum_{i=1}^{m} \sum_{j=1}^{n} {a_{ij}}^2}\,.$$

p-norms

The p-norm of a matrix is deiend in terms of the vector p-norm[2] with the following formula

$$\|A\|_p = \sup_{x \neq 0} \frac{\|A\,\underline{x}\|_p}{\|\underline{x}\|_p},$$

where the 'sup' refers to the maximum value that the norm can take.

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[1] Matrix Definitions
[2] Vector Norm

If the eigenvalues[3] of A^*A (where A^* is the conjugate transpose of A) are known and the maximum absolute eigenvalue of A^*A is $\lambda_{\max}$ then $\|A\|_2 = \sqrt{|\lambda_{\max}|}$.

<u>Example of a 2 × 2 matrix</u>

For matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$,

$$\|A\|_1 = \max(|3| + |-1|, |-2| + |4|) = \max(4,6) = 6\,,$$

$$\|A\|_\infty = \max(|3| + |-2|, |-1| + |4|) = \max(5,5) = 5\,,$$

$$\|A\|_F = \sqrt[2]{3^2 + (-2)^2 + (-1)^2 + 4^2} = \sqrt{30} = 5.4772 \text{ to four decimal places.}$$

In order to find the 2-norm we first multiply A by its transpose

$$A^*A = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix}.$$

The eigenvalues of A^*A are the solutions of $\begin{vmatrix} 10-\lambda & -10 \\ -10 & 20-\lambda \end{vmatrix} = 0$.

That is the solutions of $(10-\lambda)(20-\lambda) - 100 = 0$ or $\lambda^2 - 30\lambda + 100 = 0$.

This has the solutions $\lambda = \frac{30 \pm \sqrt{30^2 - 4\times 100}}{2} = 15 \pm 5\sqrt{5}$.

The maximum eigenvalue of A^*A is $15 + 5\sqrt{5}$ and hence the 2-norm of A is $\sqrt{15 + 5\sqrt{5}}$ =5.1167 to four decimal places.

Use and Computation of Norms

The evaluation of a matrix norm is useful whenever we need an estimate of its size. If the matix norm needs to be evaluated then the choice of norm may be dictated by the circumstances. However, in the case that the choice of norm is arbitrary then there are clear variations in the computational effort involved; the 1-norm and the ∞-norm are both fairly straightforward (requiring $O(n^2)$ additions and $O(n)$ comparisons), the Frobenius norm is also relatively straightforward, requiring $O(n^2)$ multiplications and additions and one squareroot evaluation. However, the 2-norm is quite a long-winded computation, and the p-norms are even more time consuming to compute. Hence where computer time is at a premium, and the choice of norm is not critical, the 1, ∞ and Frobenius norms are advised.

[3] Matrix Eigenvalues and Eigenvectors